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EVOLUTION OF SPACELIKE SURFACES IN ANTI-DE SITTER SPACE BY THEIR LAGRANGIAN ANGLE

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ABSTRACT. We study spacelike hypersurfaces in anti-De Sitter spacetime that evolve by the Lagrangian angle of their Gauß maps.

1. INTRODUCTION

In Lorentzian manifolds spacelike hypersurfaces of prescribed curvature (mean, scalar, Gauß etc.) are of great interest. To prove existence of such hypersurfaces, many authors use either elliptic or parabolic methods. E.g. in [E], [E2], [EH] mean curvature flow and variants of it have been used to generate spacelike hypersurfaces of prescribed mean curvature.

One advantage of the parabolic method is, that one can often prove existence of solutions merely under reasonable geometric assumptions on the initial hypersurface - imposed in terms of algebraic expressions involving the second fundamental form - and e.g. the existence of barriers might not be needed. In this paper we will see that there exists another interesting geometric flow for spacelike hypersurfaces in anti-de Sitter spacetime, where the Gauß maps of the hypersurfaces move by the Lagrangian mean curvature flow. Our aim is to prove the following two theorems:

Theorem 1.1. *Let (N,g) be a time-oriented Lorentzian manifold of signature $(n,1)$ and of constant sectional curvature $-\kappa<0$. Suppose $F_0:M\to N$ is a smooth spacelike immersion of a closed n-dimensional manifold M. There exists $T>0$ and a smooth solution $F:M\times[0,T)\to N$ of the evolution equation*

$$\begin{aligned} \frac{d}{dt}F(p,t) &= \phi(p,t)\nu(p,t)\,, \quad \forall p\in M,\, \forall t\in[0,T)\,, \qquad (1)\\ F(p,0) &= F_0(p)\,, \quad \forall p\in M\,, \end{aligned}$$

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where $\nu(p,t)$ denotes the future directed timelike unit normal at $F(p,t)$ and $\phi : M \to \mathbb{R}$ is the function

$$\phi = \frac{1}{\sqrt{\kappa}} \sum_{i=1}^{n} \arctan \frac{\lambda_i}{\sqrt{\kappa}} \tag{2}$$

with $\lambda_1 \leq \cdots \leq \lambda_n$ denoting the principal curvatures of the immersion. In particular, despite the fact that $\lambda = (\lambda_1, \ldots, \lambda_n) : M \to \mathbb{R}^n$ is merely continuous, the function ϕ is smooth.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1 assume in addition $n = 2$ and that the Gauß curvature $K = \lambda_1 \lambda_2$ of the initial surface satisfies*

$$|K| < \kappa\,. \tag{3}$$

Then this condition is preserved during the flow, a smooth solution of (1) exists for all $t \in [0, \infty)$ and F converges smoothly and exponentially to a spacelike limit surface with vanishing mean curvature $H = \lambda_1 + \lambda_2 = 0$ and with $-\kappa \leq K < \kappa$ as $t \to \infty$.

Remark 1.3. *We remark that the condition $|K| < \kappa$ implies that the two components of the associated Gauß map $\mathscr{G} : M \to Gr_2^+(2,4) = \mathbb{H}^2_{1/\sqrt{2}} \times \mathbb{H}^2_{1/\sqrt{2}}$ (see below) are immersions. Another interpretation can be given in terms of the two-positivity of the tensor $S_{ij} = \kappa g_{ij} - h_i^{\ l} h_{lj}$ w.r.t. the metric $\sigma_{ij} := \kappa g_{ij} + h_i^{\ l} h_{lj}$. where g_{ij} resp. h_{ij} denote the first resp. second fundamental tensors of F. For some geometric evolution equations two-positivity can be preserved, e.g. this has been shown in* [TW].

Definition 1.4. The function ϕ defined in equation (2) will be called the Lagrangian angle.

The last definition and also the flow defined in (1) is motivated by the following observation: Let us consider the anti-De Sitter space AdS_3 as the standard model of a Lorentzian space form with constant negative sectional curvature -1 represented by the hypersurface

$$\mathrm{AdS}_3 = \{V \in \mathbb{R}^4_2 : \langle V, V \rangle_{2,4} = -1\}$$

and equipped with the induced Lorentzian metric, where $\mathbb{R}^4_2$ denotes $\mathbb{R}^4$ with its pseudo-Riemannian metric

$$\langle V, W \rangle_{2,4} = V^1 W^1 + V^2 W^2 - V^3 W^3 - V^4 W^4\,.$$

The Gauß map of a spacelike surface $M \subset \mathrm{AdS}_3 \subset \mathbb{R}^4_2$ can be considered as a map $\mathscr{G} : M \to Gr_2^+(2,4)$, where $Gr_2^+(2,4)$ denotes the Grassmannian of oriented spacelike planes in $\mathbb{R}^4_2$. Moreover, up to scaling

$Gr_2^+(2,4)$ is isometric to $\mathbb{H}^2 \times \mathbb{H}^2$. By results of Torralbo and Urbano [T], [TU], the Gauß maps are Lagrangian. In the appendix (Lemma 5.2) we will show that ϕ is the Lagrangian angle of the Gauß map and we will also prove (Lemma 5.3) that the Gauß maps of spacelike surfaces M in AdS_3 moving by (1) will essentially, i.e. up to some tangential deformations, evolve by the Lagrangian mean curvature flow.

The flow defined by (1) has been treated in some Riemannian cases, i.e. when (N, g) is a Riemannian manifold. Andrews [A] studied the deformation of surfaces in S^3 by flows that allow an optimal control of the Gauß curvature and he detected an optimal flow with the same driving term ϕ as defined in equation (2), where in his case λ_1, λ_2 are the two principal curvatures of the surface $M \subset S^3$. In the same paper the following was mentioned without proof: If one considers $M \subset S^3 \subset \mathbb{R}^4$ as a submanifold of $\mathbb{R}^4$, then the Gauß maps $\mathscr{G} : M \to Gr(2,4)$ of M into the Grassmannian $Gr(2,4)$ of 2-planes in $\mathbb{R}^4$ will evolve by the mean curvature flow. We remark that a detailed analysis will actually show that this holds only up to tangential deformations of the image in $Gr(2,4)$ (compare also with Lemma 5.3 and with the computations in the appendix). On the other hand, Castro and Urbano [CU] proved that the Gauß maps $\mathscr{G} : M \to Gr(2,4)$ of surfaces $M \subset S^3$ are Lagrangian. Combining the results of Andrews and Castro, Urbano we see that an evolution of surfaces $M \subset S^3$ by the function $\phi = \arctan\lambda_1 + \arctan\lambda_2$ will induce (at least up to tangential deformations) a Lagrangian mean curvature flow of their Gauß maps $\mathscr{G} : M \to Gr(2,4)$. In analogy to Lemma 5.2 one can also show that ϕ is the Lagrangian angle of the Gauß map, i.e. the mean curvature 1-form τ of the Gauß map satisfies $\tau = d\phi$.

In another case, if $\theta = du$ is an exact 1-form on a flat manifold M and the graph of du considered as a submanifold in the cotangent bundle $N := T^*M$ (equipped with the flat metric) evolves by the Lagrangian mean curvature flow, then the potential u evolves by

$$\frac{\partial}{\partial t}u = \sum_{i=1}^{n} \arctan\lambda_i\,,$$

where λ_i are the eigenvalues of the Hessian D^2u and D denotes the flat connection. In particular, $\phi = \sum_{i=1}^n \arctan\lambda_i$ is again the Lagrangian angle. For details see [SW] and [S2]. Recently the Lagrangian mean curvature flow has been generalized to the case of Lagrangian submanifolds in cotangent bundles T^*M of Riemannian manifolds (M, g). If the Lagrangian submanifold can be represented as the graph of a closed

1-form $\theta \in \Omega^1(M)$, then there exists a generalized Lagrangian angle similar to the function ϕ defined above, where now λ_k, $k = 1, \ldots, n$ are the eigenvalues of the symmetric tensor $D\theta$, D denoting the Levi-Civita connection of the metric on M. For details see [SW2].

The organization of this paper is as follows: In section 2 we will introduce our notation and briefly recall some of the geometry in Lorentzian manifolds of constant sectional curvature. In section 3 we will first study arbitrary variations of spacelike hypersurfaces in Lorentzian manifolds of constant sectional curvature $-\kappa < 0$ and we will then characterize the flow defined in (1) as optimal w.r.t. the reaction term in the evolution equation of the driving function (Lemma 3.5). In this section we will also prove the smoothness of ϕ and Theorem 1.1. Section 4 is completely devoted to the two-dimensional case. We will first prove uniform C^2- and C^1-estimates and can then establish the proof of Theorem 1.2. In the appendix we will explain the relation between ϕ and the Gauß map of spacelike surfaces in AdS_3 and we show that the Gauß maps will evolve under the Lagrangian mean curvature flow, if the spacelike surfaces in AdS_3 evolve by (1).

2. Geometry of spacelike hypersurfaces in Lorentzian space forms

In this section we recall some basic facts concerning the geometry of time-oriented Lorentzian manifolds (N, g) of signature $(n, 1)$ and of constant sectional curvature $-\kappa < 0$. If $\kappa = 1$ and N is complete, then (N, g) is called a complete anti-De Sitter structure. Of particular interest in this paper will be the three-dimensional case. By results of Kulkarni, Raymond [KR] and Goldman [G2] we know that closed 3-manifolds with a complete anti-De Sitter structure are necessarily orientable Seifert fibre spaces with nonzero Euler number and with hyperbolic base. In a celebrated paper by Mess [M] (which despite its great influence it had, was unpublished until recently; see also the "Notes on Mess' paper" [ABB$^+$], published in the same volume), maximal globally hyperbolic Cauchy-compact spacetimes (called "domains of dependence") of constant curvature in $2+1$ dimensions were studied. Recall that a globally hyperbolic (Cauchy-compact) spacetime N is a spacetime admitting a (compact) spacelike hypersurface M such that every inextendable timelike curve intersects M exactly once and such that the order relation is given by the existence of isometric embeddings. Mess gave a classification in the flat and anti-De Sitter cases.

The De Sitter case was also studied by Mess but a classification was obtained later by Scannell [S]. In the anti-De Sitter case, domains of dependence are quotients of convex open sets of the anti-De Sitter space, by discrete groups of isometries. Mess exhibited a one-to-one correspondence between anti-De Sitter domains of dependence and pairs of points in the Teichmüller space. This result has been extended to $2+1$-dimensional anti-De Sitter domains of dependence having only a complete Cauchy surface by Barbot [B] and Benedetti and Bonsante [BB]. Locally, any Lorentz three-manifold with complete anti-De Sitter structure is isometric to the classical model AdS_3 (or likewise to its simply connected universal cover).

Suppose now that $F : M \to N$ is a smooth spacelike immersion of an n-dimensional oriented manifold M into a time-oriented spacetime of constant sectional curvature $-\kappa$, $\kappa > 0$. To describe the geometry of (M, F^*g) and (N, g) we will often use local coordinate systems (U, x, Ω) and (V, y, Λ) for M resp. N where we assume here and in the following:

i) $U \subset M$ is an open set around some point $p \in M$ and $x : U \to \Omega$ is a diffeomorphism between U and some open set $\Omega \subset \mathbb{R}^n$.
ii) $V \subset N$ is an open set around the point $q := F(p) \in N$ and $y : V \to \Lambda$ is a diffeomorphism between V and some open set $\Lambda \subset \mathbb{R}^{n+1}$.
iii) The coordinate systems are always chosen in such a way that $F(U) \subset V$.
iv) For the set of coordinates on M we will use Latin indices, i.e. $x = (x^i)_{i=1,\dots,n}$. Similarly, we will use Greek indices for the coordinates on N, i.e. $y = (y^\alpha)_{\alpha=1,\dots,n+1}$.

In these local coordinates, geometric quantities on M and N will then often be distinguished simply by use of Latin or Greek indices, e.g. $g = g_{\alpha\beta} dy^\alpha \otimes dy^\beta$ and $F^*g = g_{ij} dx^i \otimes dx^j$ will denote the Lorentzian resp. the induced Riemannian metric tensors on N resp. M. We also define $F^\alpha(x) := (y^\alpha \circ F)(x)$ and using the Einstein summation convention we have

$$g_{ij} = g_{\alpha\beta} F_i^\alpha F_j^\beta \, ,$$

where $F_i^\alpha := \partial F^\alpha / \partial x^i$. Let ∇ denote the Levi-Civita connection on (M, F^*g). The second fundamental tensor A is by definition $A = \nabla dF$, where

$$dF = F_i^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \in \Gamma(F^*TN \otimes T^*M)$$

is the differential of F. The second fundamental tensor $h = h_{ij}dx^i \otimes dx^j$ w.r.t. the future directed timelike unit normal $\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha}$ along $F(M)$ is given by $h = -g(A, \nu)$ and can be expressed in local coordinates by Gauß' formula

$$\nabla_i F_j^\alpha = A_{ij}^\alpha = h_{ij}\nu^\alpha = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x^k} - \Gamma_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\,, \tag{4}$$

where Γ_{ij}^k and $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of g_{ij} resp. $g_{\alpha\beta}$. The principal curvatures $\lambda_1, \ldots, \lambda_n$ at a point $p \in M$ are the eigenvalues of the Weingarten map

$$\mathscr{W} : T_pM \to T_pM\,, \quad \mathscr{W}(V) = \nabla_V \nu$$

which in local coordinates applied to a vector $V = V^i \frac{\partial}{\partial x^i}$ is given by

$$\mathscr{W}V = \mathscr{W}_i^k V^i \frac{\partial}{\partial x^k}\,, \quad \mathscr{W}_i^k = h^k{}_i = g^{kl}h_{li}\,,$$

where $(g^{kl})_{k,l=1,\ldots,n}$ denotes the inverse of $(g_{kl})_{k,l=1,\ldots,n}$ and where indices will be raised and lowered w.r.t. the metric tensors g_{ij} resp. g^{ij}.

Two other important relations are given by the Codazzi equation

$$\nabla_i h_{jk} = \nabla_j h_{ik} \tag{5}$$

and the Gauß' equation which in view of the constancy of the sectional curvatures on (N, g) is

$$R_{ijkl} = -\kappa(g_{ik}g_{jl} - g_{il}g_{jk}) - h_{ik}h_{jl} + h_{il}h_{jk}\,. \tag{6}$$

Here R_{ijkl} is the Riemannian curvature tensor w.r.t. the metric g_{ij} on M. Using the rule for interchanging derivatives together with (5) and (6) we compute

$$\begin{aligned} \nabla_i\nabla_j h_{kl} &= \nabla_i \nabla_k h_{lj} \\ &= \nabla_k\nabla_i h_{lj} + R^m{}_{lki}h_{mj} + R^m{}_{jki}h_{lm} \\ &= \nabla_k\nabla_l h_{ij} + R^m{}_{lki}h_{mj} + R^m{}_{jki}h_{lm}\,. \end{aligned} \tag{7}$$

We will use (7) in the sequel (the trace is known as Simons' identity).

3. Variations of spacelike hypersurfaces in Lorentzian space forms

In this section assume that for some $T > 0$ we are given a smooth family of spacelike immersions

$$F : M \times [0, T) \to N$$

such that

$$\frac{d}{dt}\,F(p,t)=f(p,t)\nu(p,t)\,,\quad \forall p\in M,\,\forall\, t\in[0,T)\,, \tag{8}$$

where $f(\cdot,t)$ is a smooth function depending smoothly on the principal curvatures $\lambda_1,\ldots,\lambda_n$ of the immersed hypersurface $M_t:=F(M,t)$, and where $\nu(p,t)$ is the future directed timelike unit normal at $F(p,t)$.

The evolution equations for the first and second fundamental forms are described in the next Lemma, a proof of which can be found in the literature (c.f. [G], see also [EH])

Lemma 3.1 (Evolution equations)**.** *The evolution equations of the first and second fundamental form of spacelike hypersurfaces in Lorentzian manifolds of constant sectional curvature $-\kappa$ evolving by (8) are*

$$\frac{d}{dt}\,g_{ij} = 2fh_{ij}\,, \tag{9}$$

$$\frac{d}{dt}\,h_{ij} = \nabla_i\nabla_j f + fh_i^{\ k}h_{jk} - \kappa f g_{ij}\,. \tag{10}$$

For a smooth function f depending on the eigenvalues $(\lambda_i)_{i=1,\ldots,n}$ of the second fundamental form let us define the tensors

$$f^{ij}:=\frac{\partial f}{\partial h_{ij}}\,,\quad f^{ij,kl}:=\frac{\partial^2 f}{\partial h_{kl}\partial h_{ij}}\,.$$

Then

$$\begin{aligned}\nabla_i\nabla_j f &= \nabla_i\left(f^{kl}\nabla_j h_{kl}\right)\\ &= f^{kl}\nabla_i\nabla_j h_{kl} + f^{kl,pq}\nabla_i h_{pq}\nabla_j h_{kl}\,.\end{aligned}$$

If we insert this into (10) and use (6), (7), then we get

$$\begin{aligned}\frac{d}{dt}\,h_{ij} &= f^{kl}\nabla_k\nabla_l h_{ij} + f^{kl,pq}\nabla_i h_{pq}\nabla_j h_{kl}\\ &\quad -\kappa f^{kl}(h_{jk}g_{il} - h_{ij}g_{kl} + h_{kl}g_{ij} - h_{il}g_{jk})\\ &\quad +f^{kl}(-h_j^{\ m}h_{mk}h_{il} + h_i^{\ m}h_{mj}h_{kl} - h_k^{\ m}h_{ml}h_{ij} + h_i^{\ m}h_{ml}h_{jk})\\ &\quad +fh_i^{\ k}h_{jk} - \kappa f g_{ij}\,. \end{aligned} \tag{11}$$

If G is another smooth function that depends smoothly on the principal curvatures, we may consider G as a function of the tensors h_{ij} and g^{ij}, so that for example

$$\begin{aligned}\nabla_j G &= \frac{\partial G}{\partial h_{kl}}\nabla_j h_{kl} + \frac{\partial G}{\partial g^{kl}}\nabla_j g^{kl}\\ &= \frac{\partial G}{\partial h_{kl}}\nabla_j h_{kl}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\nabla_i\nabla_j G &= \frac{\partial^2 G}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial G}{\partial x^k}\\ &= \frac{\partial G}{\partial h_{kl}}\nabla_i\nabla_j h_{kl} + \frac{\partial^2 G}{\partial h_{kl}\partial h_{pq}}\nabla_i h_{kl}\nabla_j h_{pq}\,. \qquad (13)\end{aligned}$$

Likewise

$$\frac{d}{dt}G = \frac{\partial G}{\partial h_{ij}}\frac{d}{dt}h_{ij} + \frac{\partial G}{\partial g^{ij}}\frac{d}{dt}g^{ij}\,. \qquad (14)$$

Let us define

$$G^{kl} := \frac{\partial G}{\partial h_{kl}}\,, \quad G^{kl,pq} := \frac{\partial^2 G}{\partial h_{pq}\partial h_{kl}}\,.$$

Then combining (9) and (10) one gets

Lemma 3.2. *Under the flow (8) the evolution equation of an arbitrary smooth function G, depending smoothly on the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the Weingarten map $\mathscr{W}$ is*

$$\begin{aligned}\frac{d}{dt}G - f^{ij}\nabla_i\nabla_j G &= G^{ij}f^{kl}(\nabla_i\nabla_j h_{kl} - \nabla_k\nabla_l h_{ij})\\ &\quad + \left(G^{ij}f^{pq,kl} - f^{ij}G^{pq,kl}\right)\nabla_i h_{kl}\nabla_j h_{pq}\\ &\quad + f\left(G^{ij}(h_{il}h^l{}_j - \kappa g_{ij}) - 2\frac{\partial G}{\partial g^{kl}}h^{kl}\right)\,. \qquad (15)\end{aligned}$$

We observe that for $G = f$ the first two lines on the RHS vanish so that we obtain as a corollary:

Corollary 3.3. *Under the flow (8) the evolution equation of f itself is*

$$\frac{d}{dt}f = f^{ij}\nabla_i\nabla_j f + f\left(f^{ij}(h_{il}h^l{}_j - \kappa g_{ij}) - 2\frac{\partial f}{\partial g^{kl}}h^{kl}\right)\,. \qquad (16)$$

So far, f is arbitrary. If we want the flow to be parabolic, then we must assume that the tensor f^{ij} is positive definite. Our idea is to choose a function f in such a way that the flow is parabolic and such that the reaction terms on the RHS of (16) simplify as much as reasonable, e.g. so that the term in the brackets is constant.

To make an ansatz, we first consider the case $n = 1$, so that f merely depends on the (mean) curvature $\lambda = g^{kl}h_{kl}$. We then obtain

$$f^{kl} = f' g^{kl}\,, \quad \frac{\partial f}{\partial g^{kl}} = f' h_{kl}\,,$$

where $f' = \partial f/\partial\lambda$. Hence

$$f^{ij}(h_{il}h^l{}_j - \kappa g_{ij}) - 2\frac{\partial f}{\partial g^{kl}}h^{kl} = -f'(\kappa + \lambda^2)\,.$$

The flow is parabolic if $f' > 0$. Thus we are looking for a monotone increasing (in λ) function f for which

$$f'(\kappa + \lambda^2) = c$$

for some constant c. But $\kappa > 0$, $f' > 0$ and $\kappa + \lambda^2 > 0$ imply $c > 0$ and then

$$f(\lambda) = \frac{c}{\sqrt{\kappa}} \arctan \frac{\lambda}{\sqrt{\kappa}} + a\,,$$

with some arbitrary constants a and $c > 0$. We will choose $c = 1$ and $a = 0$. For general n we now simply sum over all eigenvalues of the Weingarten map and thus define

$$f(p) := \phi(p) = \frac{1}{\sqrt{\kappa}} \sum_{j=1}^{n} \arctan \frac{\lambda_j(p)}{\sqrt{\kappa}}\,. \tag{17}$$

For each k the function λ_k is continuous but in general not smooth. Surprisingly, as the next Lemma shows, the function ϕ in (17) is smooth.

Lemma 3.4. *Suppose $F : M \to (N, g)$ is a smooth spacelike immersion into a time-oriented Lorentzian manifold of constant sectional curvature $-\kappa < 0$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ denote the principal curvature functions on M. The function $\phi : M \to \mathbb{R}$ defined in (17) is smooth.*

Proof. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n : M \to \mathbb{R}$ be the principal curvature functions on M. It is well-known that each λ_k is continuous but in general not smooth. Hence ϕ is at least continuous. W.l.o.g. we may assume $\kappa = 1$. On $\mathbb{R}^n$ we consider the following three smooth functions:

$$\alpha : \mathbb{R}^n \to \left(-\frac{n\pi}{2}, \frac{n\pi}{2}\right)\,, \quad \alpha(x_1, \ldots, x_n) = \sum_{k=1}^{n} \arctan x_k\,,$$

$$a : \mathbb{R}^n \to \mathbb{R}\,, \quad a(x_1, \ldots, x_n) := \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k s_{2k+1}(x_1, \ldots, x_n)\,,$$

$$b : \mathbb{R}^n \to \mathbb{R}\,, \quad b(x_1, \ldots, x_n) := \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k s_{2k}(x_1, \ldots, x_n)\,,$$

where s_k are the elementary symmetric functions, e.g.

$$s_0(x_1, \ldots, x_n) = 1\,, \quad s_1(x_1, \ldots, x_n) = \sum_{k=1}^n x_k$$

and

$$s_n(x_1, \ldots, x_n) = \prod_{k=1}^n x_k\,.$$

By induction one can show

$$a^2(x_1, \ldots, x_n) + b^2(x_1, \ldots, x_n) = \prod_{k=1}^n (1 + x_k^2) \geq 1\,,$$

so that a and b cannot vanish simultaneously in a point $x \in \mathbb{R}^n$. Let $U_a := \{x \in \mathbb{R}^n : a(x) \neq 0\}$ and $U_b := \{x \in \mathbb{R}^n : b(x) \neq 0\}$. Then $U_a \cup U_b = \mathbb{R}^n$ and both U_a and U_b are open. We obtain two smooth functions

$$\alpha_a : U_a \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\,, \quad \alpha_a(x) := -\arctan\frac{b(x)}{a(x)}\,,$$

$$\alpha_b : U_b \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\,, \quad \alpha_b(x) := \arctan\frac{a(x)}{b(x)}\,.$$

Now at each $x \in U_a$ we have

$$\frac{\partial \alpha_a}{\partial x_k} = \frac{1}{1 + x_k^2} = \frac{\partial \alpha}{\partial x_k}\,,$$

and likewise at any $x \in U_b$

$$\frac{\partial \alpha_b}{\partial x_k} = \frac{1}{1 + x_k^2} = \frac{\partial \alpha}{\partial x_k}\,,$$

so that $\alpha_a - \alpha$ resp. $\alpha_b - \alpha$ are constant on each connected component of U_a resp. U_b. We are now ready to prove the smoothness of ϕ. Let $p \in M$ be arbitrary. At p we must either have $a(\lambda_1(p), \ldots, \lambda_n(p)) \neq 0$ or $b(\lambda_1(p), \ldots, \lambda_n(p)) \neq 0$. W.l.o.g. assume $b(\lambda(p)) \neq 0$ (the other case can be treated similarly). Since the elementary symmetric functions

$$\tilde{s}_k : M \to \mathbb{R}\,, \quad p \mapsto s_k(\lambda_1(p), \ldots, \lambda_n(p))$$

are smooth ($\tilde{s}_0 = 1$, $\tilde{s}_1 = H, \ldots, \tilde{s}_n = K$), we know that $\tilde{a} := a \circ \lambda$ and $\tilde{b} := b \circ \lambda$ are smooth functions on all of M since they can be expressed in terms of the elementary symmetric functions. Choose a small open neighborhood $U \subset M$ around p with $b(\lambda(q)) \neq 0$ for all $q \in U$ and such that $\lambda(q)$ lies in the same connected component of U_b for any $q \in U$ (the latter works due to the continuity of λ). Then $\phi = \alpha \circ \lambda$ implies that $\phi - \arctan\frac{\tilde{a}}{\tilde{b}}$ is a constant function on U. Since $\arctan\frac{\tilde{a}}{\tilde{b}}$ is smooth

on U, so must be ϕ. This proves the claim.

$\square$

Let us define

$$\sigma_{ij} := \kappa g_{ij} + h_i^{\ l} h_{lj}\,. \tag{18}$$

From the construction of ϕ we get

$$\frac{\partial \phi}{\partial h_{kl}} = \sigma^{kl}\,, \tag{19}$$

$$\frac{\partial \phi}{\partial g^{kl}} = g_{ik} h_{jl} \sigma^{ij}\,, \tag{20}$$

$$\phi^{pq,kl} = \frac{\partial \sigma^{pq}}{\partial h_{kl}} = -(\sigma^{pk}\sigma^{qj} + \sigma^{qk}\sigma^{pj}) h^l_{\ j}\,. \tag{21}$$

where σ^{ij} shall denote the inverse of σ_{ij}.

(22)

Proof of Theorem 1.1: By Lemma 3.4 the function ϕ is smooth and since $\sigma^{ij} = \frac{\partial \phi}{\partial h_{ij}}$ is positive definite, the flow defined by (1) is parabolic. The statement now follows from the standard theory of parabolic evolution equations on smooth compact manifolds.

$\square$

Applying (19) and (20) to the general evolution equation (16) in case $f = \phi$ we obtain

Lemma 3.5. *Under the flow given by equation (1) we have*

$$\frac{d}{dt}\,\phi = \sigma^{ij} \nabla_i \nabla_j \phi - n\phi\,. \tag{23}$$

Proof. This follows directly from the construction of ϕ and likewise from equations (19), (20):

$$\begin{aligned} f^{ij}(h_{il}h^l_{\ j} - \kappa g_{ij}) - 2\frac{\partial f}{\partial g^{kl}} h^{kl} &= \sigma^{ij}(h_{il}h^l_{\ j} - \kappa g_{ij}) - 2 g_{ik} h_{jl} \sigma^{ij} h^{kl} \\ &= -\sigma^{ij}(\kappa g_{ij} + h_{il} h^l_{\ j}) \\ &= -\sigma^{ij}\sigma_{ij} = -n\,. \end{aligned}$$

$\square$

A direct consequence of (23) and the maximum principle gives:

Lemma 3.6. *Under the flow given by equation (1) the Lagrangian angle ϕ satisfies the estimate:*

$$\inf_{q\in M}\phi(q,0)\le\phi(p,t)e^{nt}\le\sup_{q\in M}\phi(q,0)\,,\quad\forall(p,t)\in M\times[0,T)\,.$$

Note that the quantity $g_{ik}h_{jl}\sigma^{ij}$ is symmetric in k and l. More generally, for any non-negative integer r we define a tensor $\overset{(r)}{h}_{is}$ by

$$\overset{(r)}{h}_{is}:=\begin{cases}g_{is} & ,r=0\\ h_{is} & ,r=1\\ \overset{(r-1)}{h}_{ij}h^j{}_s & ,r\ge 2\,.\end{cases}$$

Then we have

Lemma 3.7. *For all integers $r,s\ge 0$ the following symmetry holds:*

$$\sigma^{kl}\overset{(r)}{h}_{ki}\overset{(s)}{h}_{lj}=\sigma^{kl}\overset{(r)}{h}_{kj}\overset{(s)}{h}_{li}\,. \tag{24}$$

Proof. We choose an orthonormal basis $e_1,\dots,e_n$ at a point $p\in M$ so that h_{ij} becomes diagonal at p, i.e.

$$h_{ij}=\operatorname{diag}(\lambda_1,\dots,\lambda_n)\,.$$

Then all tensors $\overset{(k)}{h}_{ij}$ become diagonal at p as well, more precisely

$$\overset{(k)}{h}_{ij}=\operatorname{diag}(\lambda_1^k,\dots,\lambda_n^k)\,.$$

In addition we have at p

$$\sigma_{ij}=\operatorname{diag}(\kappa+\lambda_1^2,\dots,\kappa+\lambda_n^2)\,,\quad\sigma^{ij}=\operatorname{diag}\left(\frac{1}{\kappa+\lambda_1^2},\dots,\frac{1}{\kappa+\lambda_n^2}\right).$$

This implies the symmetries. □

4. The two-dimensional case

In the two-dimensional case we are able to prove a longtime existence and convergence result under the assumption that the Gauß curvature K of the spacelike surface is strictly bounded by

$$-\kappa<K<\kappa.$$

Therefore, in this section let us assume $n=2$ and let $K=\lambda_1\lambda_2$ be the Gauß curvature and $H=\lambda_1+\lambda_2$ the mean curvature.

From $K=\frac{1}{2}(H^2-|h|^2)$, where $|h|^2=\lambda_1^2+\lambda_2^2$ is the squared norm of the second fundamental form, and from $n=2$ we get

$$\overset{(2)}{h}_{ij}=Hh_{ij}-Kg_{ij}\,. \tag{25}$$

An easy computation yields

$$\begin{aligned}
\sigma^{ij} &= \frac{1}{\kappa H^2 + (\kappa - K)^2}\Big((2\kappa + |h|^2)g^{ij} - \kappa g^{ij} - g^{ik}g^{jl}h^{(2)}_{kl}\Big) \\
&= \frac{1}{\kappa H^2 + (\kappa - K)^2}\big((2(\kappa - K) + H^2)g^{ij} - \kappa g^{ij} - Hh^{ij} + Kg^{ij}\big) \\
&= \frac{1}{\kappa H^2 + (\kappa - K)^2}\big((\kappa - K + H^2)g^{ij} - Hh^{ij}\big)\ . \qquad (26)
\end{aligned}$$

Moreover we compute

$$\frac{\partial H}{\partial h_{ij}} = g^{ij}\,, \quad \frac{\partial K}{\partial h_{ij}} = Hg^{ij} - h^{ij} \qquad (27)$$

and

$$\frac{\partial H}{\partial g^{ij}} = h_{ij}\,, \quad \frac{\partial K}{\partial g^{ij}} = Hh_{ij} - h^{(2)}_{ij} = Kg_{ij}\,. \qquad (28)$$

Like in the previous sections, let G denote an arbitrary function depending smoothly on the eigenvalues λ_1, λ_2 of the Weingarten map. Since λ_1 and λ_2 can be computed from H and K, we may assume that G depends only on H and K. We set

$$G_H := \frac{\partial G}{\partial H}\,, \quad G_K := \frac{\partial G}{\partial K}\,.$$

From (27) and (28) we conclude

$$\begin{aligned}
G^{ij} = \frac{\partial G}{\partial h_{ij}} &= G_H\frac{\partial H}{\partial h_{ij}} + G_K\frac{\partial K}{\partial h_{ij}} \\
&= G_H g^{ij} + G_K(Hg^{ij} - h^{ij}) \\
&= (G_H + HG_K)g^{ij} - G_K h^{ij} \qquad (29)
\end{aligned}$$

and

$$\frac{\partial G}{\partial g^{ij}} = KG_K g_{ij} + G_H h_{ij}\,. \qquad (30)$$

From the general evolution equation of G given by equation (15) we now derive

$$\begin{aligned}
\frac{d}{dt}G - \sigma^{ij}\nabla_i\nabla_j G &= \big(G^{ij}\sigma^{pq,kl} - \sigma^{ij}G^{pq,kl}\big)\nabla_i h_{kl}\nabla_j h_{pq} \\
&\quad + G^{ij}\sigma^{kl}(\nabla_i\nabla_j h_{kl} - \nabla_k\nabla_l h_{ij}) \\
&\quad + f\left(G^{ij}(h^{(2)}_{ij} - \kappa g_{ij}) - 2\frac{\partial G}{\partial g^{kl}}h^{kl}\right)\ .
\end{aligned}$$

We first simplify the last term

$$
\begin{aligned}
& G^{ij}(h^{(2)}_{ij} - \kappa g_{ij}) - 2\frac{\partial G}{\partial g^{kl}} h^{kl} \\
= & \ \big((G_H + HG_K)g^{ij} - G_K h^{ij}\big)\big(Hh_{ij} - (\kappa + K)g_{ij}\big) - 2(KG_K g_{ij} + G_H h_{ij})h^{ij} \\
= & \ (G_H + HG_K)(H^2 - 2(\kappa + K)) - HG_K|h|^2 + H(\kappa + K)G_K - 2HKG_K - 2|h|^2 G_H \\
= & \ G_H(H^2 - 2(\kappa + K) - 2|h|^2) + HG_K(H^2 - 2(\kappa + K) - |h|^2 + \kappa + K - 2K) \\
= & \ -(H^2 + 2(\kappa - K))G_H - H(\kappa + K)G_K \qquad (31)
\end{aligned}
$$

Let us also compute the second term:

$$
\begin{aligned}
& G^{ij}\sigma^{kl}(\nabla_i\nabla_j h_{kl} - \nabla_k\nabla_l h_{ij}) \\
= & \ -G^{ij}\sigma^{kl}(R_{mlik}h^m{}_j + R_{mjik}h^m{}_l) \\
= & \ (\kappa + K)G^{ij}\sigma^{kl}(h_{ij}g_{kl} - h_{kj}g_{li} + h_{il}g_{jk} - h_{kl}g_{ij}) \\
= & \ (\kappa + K)(G^{ij}h_{ij}\sigma^{kl}g_{kl} - G^{ij}g_{ij}\sigma^{kl}h_{kl})\,.
\end{aligned}
$$

On the other hand we compute

$$
\begin{aligned}
G^{ij}h_{ij} &= \big[(G_H + HG_K)g^{ij} - G_K h^{ij}\big]h_{ij} \\
&= HG_H + 2KG_K\,,
\end{aligned}
$$

$$
\begin{aligned}
G^{ij}g_{ij} &= \big[(G_H + HG_K)g^{ij} - G_K h^{ij}\big]g_{ij} \\
&= 2G_H + HG_K\,,
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{kl}h_{kl} &= \frac{1}{\kappa H^2 + (\kappa - K)^2}\big((\kappa - K + H^2)g^{kl} - Hh^{kl}\big)\,h_{kl} \\
&= \frac{H(\kappa - K + H^2 - |h|^2)}{\kappa H^2 + (\kappa - K)^2} \\
&= \frac{H(\kappa + K)}{\kappa H^2 + (\kappa - K)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma^{kl}g_{kl} &= \frac{1}{\kappa H^2 + (\kappa - K)^2}\big((\kappa - K + H^2)g^{kl} - Hh^{kl}\big)\,g_{kl} \\
&= \frac{2(\kappa - K + H^2) - H^2}{\kappa H^2 + (\kappa - K)^2} \\
&= \frac{2(\kappa - K) + H^2}{\kappa H^2 + (\kappa - K)^2}
\end{aligned}
$$

so that

$$
\begin{aligned}
& G^{ij}\sigma^{kl}(\nabla_i\nabla_j h_{kl} - \nabla_k\nabla_l h_{ij}) \\
= \;& (\kappa+K)(G^{ij}h_{ij}\sigma^{kl}g_{kl} - G^{ij}g_{ij}\sigma^{kl}h_{kl}) \\
= \;& \frac{\kappa+K}{\kappa H^2+(\kappa-K)^2}\Big((HG_H+2KG_K)(2(\kappa-K)+H^2) \\
& \qquad\qquad\qquad -(2G_H+HG_K)H(\kappa+K)\Big) \\
= \;& \frac{\kappa+K}{\kappa H^2+(\kappa-K)^2}\Big(HG_H\big(2(\kappa-K)+H^2-2(\kappa+K)\big) \\
& +G_K\big(2K(2(\kappa-K)+H^2)-H^2(\kappa+K)\big)\Big) \\
= \;& \frac{(\kappa+K)(H^2-4K)}{\kappa H^2+(\kappa-K)^2}\Big(HG_H-(\kappa-K)G_K\Big)
\end{aligned}
$$

So far, combining everything we have shown:

Lemma 4.1. *Suppose M is 2-dimensional and $F : M \times [0,T) \to N$ evolves by (1). Then the evolution equation of a function G that depends smoothly on the principal curvatures is given by*

$$
\begin{aligned}
\frac{d}{dt}G - \sigma^{ij}\nabla_i\nabla_j G \;=\;& \left(G^{ij}\sigma^{pq,kl} - \sigma^{ij}G^{pq,kl}\right)\nabla_i h_{kl}\nabla_j h_{pq} \\
& + \frac{(\kappa+K)(H^2-4K)}{\kappa H^2+(\kappa-K)^2}\Big(HG_H-(\kappa-K)G_K\Big) \\
& -\phi\left((H^2+2(\kappa-K))G_H + H(\kappa+K)G_K\right) .
\end{aligned} \tag{32}
$$

The next lemma is interesting in its own right since it gives a precise relation between the full norm of ∇h and ∇H on general 2-dimensional Riemannian manifolds.

Lemma 4.2. *Let (M,g) be a 2-dimensional Riemannian manifold and suppose $h \in \Gamma(T^*M \otimes T^*M)$ is a smooth symmetric Codazzi tensor, i.e. in local coordinates we have $h = h_{ij}dx^i \otimes dx^j$ and*

$$
h_{ij} = h_{ji} \quad \forall\, i,j\,, \tag{33}
$$

$$
\nabla_i h_{jk} = \nabla_j h_{ik} \quad \forall\, i,j,k\,, \tag{34}
$$

where ∇ denotes the Levi-Civita connection of g. Let $H = g^{ij}h_{ij}$ be the trace and $\overset{\circ}{h}_{ij} := h_{ij} - H/2\, g_{ij}$ be the tracefree part of h_{ij}. Then the following identity holds:

$$
2|\overset{\circ}{h}|^2\big(|\nabla h|^2 - |\nabla H|^2\big) = |\nabla|\overset{\circ}{h}|^2|^2 - 2\overset{\circ}{h}_{ij}\nabla^i|\overset{\circ}{h}|^2\nabla^j H\,. \tag{35}
$$

Proof. Let $p \in M$ be arbitrary. At p we choose normal coordinates such that h is diagonal at p, say $h_{ij} = \operatorname{diag}(\lambda_1, \lambda_2)$. This is possible since h_{ij} is symmetric. Then we compute

$$
\begin{aligned}
|\nabla \mathring{h}|^2 &= \sum_{i,j,k=1}^{2} (\nabla_k \mathring{h}_{ij})^2 \\
&= (\nabla_1 \mathring{h}_{11})^2 + 2(\nabla_1 \mathring{h}_{12})^2 + (\nabla_2 \mathring{h}_{11})^2 \\
&\quad + (\nabla_2 \mathring{h}_{22})^2 + 2(\nabla_2 \mathring{h}_{12})^2 + (\nabla_1 \mathring{h}_{22})^2 .
\end{aligned}
$$

Since $\mathring{h}_{11} = -\mathring{h}_{22}$ we obtain $\nabla_i \mathring{h}_{11} = -\nabla_i \mathring{h}_{22}$ and then

$$
\begin{aligned}
|\nabla \mathring{h}|^2 &= 4(\nabla_1 \mathring{h}_{11})^2 + 4(\nabla_2 \mathring{h}_{22})^2 \\
&\quad + 2\left((\nabla_1 \mathring{h}_{12})^2 - (\nabla_2 \mathring{h}_{11})^2 + (\nabla_2 \mathring{h}_{12})^2 - (\nabla_1 \mathring{h}_{22})^2 \right) . \qquad (36)
\end{aligned}
$$

Next we compute

$$
\begin{aligned}
|\nabla |\mathring{h}|^2|^2 &= 4 \sum_{k=1}^{2} \left(\sum_{i,j=1}^{2} \mathring{h}_{ij} \nabla_k \mathring{h}_{ij} \right)^2 \\
&= 4 \left(\mathring{h}_{11} \nabla_1 \mathring{h}_{11} + \mathring{h}_{22} \nabla_1 \mathring{h}_{22} \right)^2 + 4 \left(\mathring{h}_{11} \nabla_2 \mathring{h}_{11} + \mathring{h}_{22} \nabla_2 \mathring{h}_{22} \right)^2 \\
&= 4 \left(2\mathring{h}_{11} \nabla_1 \mathring{h}_{11}) \right)^2 + 4 \left(-2\mathring{h}_{11} \nabla_2 \mathring{h}_{22}) \right)^2 \\
&= 16 (\mathring{h}_{11})^2 \left((\nabla_1 \mathring{h}_{11})^2 + (\nabla_2 \mathring{h}_{22})^2 \right) . \qquad (37)
\end{aligned}
$$

Combining (36), (37) and $|\mathring{h}|^2 = (\mathring{h}_{11})^2 + (\mathring{h}_{22})^2 = 2(\mathring{h}_{11})^2$ we get

$$
2|\mathring{h}|^2 \cdot |\nabla \mathring{h}|^2 - |\nabla |\mathring{h}|^2|^2 = 4|\mathring{h}|^2 \left((\nabla_1 \mathring{h}_{12})^2 - (\nabla_2 \mathring{h}_{11})^2 + (\nabla_2 \mathring{h}_{12})^2 - (\nabla_1 \mathring{h}_{22})^2 \right) . \qquad (38)
$$

From $\nabla_i h_{jk} = \nabla_j h_{ik}$ we obtain

$$
\nabla_i \mathring{h}_{jk} - \nabla_j \mathring{h}_{ik} = \frac{1}{2} \left(\nabla_j H g_{ik} - \nabla_i H g_{jk} \right)
$$

so that

$$
(\nabla_1 \mathring{h}_{12})^2 - (\nabla_2 \mathring{h}_{11})^2 = \frac{1}{2} (\nabla_1 \mathring{h}_{12} + \nabla_2 \mathring{h}_{11}) \nabla_2 H = \nabla_2 \mathring{h}_{11} \nabla_2 H + \frac{1}{4} (\nabla_2 H)^2
$$

and

$$
(\nabla_2 \mathring{h}_{12})^2 - (\nabla_1 \mathring{h}_{22})^2 = \nabla_1 \mathring{h}_{22} \nabla_1 H + \frac{1}{4} (\nabla_1 H)^2 .
$$

Then (38) implies

$$2|\overset{\circ}{h}|^2 \cdot |\nabla \overset{\circ}{h}|^2 - |\nabla |\overset{\circ}{h}|^2|^2 = |\overset{\circ}{h}|^2 \left(|\nabla H|^2 + 4\nabla_1 \overset{\circ}{h}_{22} \nabla_1 H - 4\nabla_2 \overset{\circ}{h}_{22} \nabla_2 H \right) . \tag{39}$$

In a next step we compute

$$\begin{aligned} \nabla_i |\overset{\circ}{h}|^2 &= 2\overset{\circ}{h}_{11} \nabla_i \overset{\circ}{h}_{11} + 2\overset{\circ}{h}_{22} \nabla_i \overset{\circ}{h}_{22} \\ &= -4\overset{\circ}{h}_{11} \nabla_i \overset{\circ}{h}_{22} \end{aligned} \tag{40}$$

where we have used that $|\overset{\circ}{h}|^2$ is a smooth function and $\overset{\circ}{h}_{ij}$ is diagonal and tracefree. Then we get

$$\begin{aligned} \overset{\circ}{h}_{ij} \nabla^i |\overset{\circ}{h}|^2 \nabla^j H &= \overset{\circ}{h}_{11} \nabla_1 |\overset{\circ}{h}|^2 \nabla_1 H + \overset{\circ}{h}_{22} \nabla_2 |\overset{\circ}{h}|^2 \nabla_2 H \\ &\overset{(40)}{=} -4(\overset{\circ}{h}_{11})^2 \nabla_1 \overset{\circ}{h}_{22} \nabla_1 H - 4\overset{\circ}{h}_{22} \overset{\circ}{h}_{11} \nabla_2 \overset{\circ}{h}_{22} \nabla_2 H \\ &= -(\overset{\circ}{h}_{11})^2 (4\nabla_1 \overset{\circ}{h}_{22} \nabla_1 H - 4\nabla_2 \overset{\circ}{h}_{22} \nabla_2 H) \\ &= -\frac{1}{2} |\overset{\circ}{h}|^2 (4\nabla_1 \overset{\circ}{h}_{22} \nabla_1 H - 4\nabla_2 \overset{\circ}{h}_{22} \nabla_2 H) . \end{aligned}$$

Combining with (39) we get

$$2|\overset{\circ}{h}|^2 \cdot |\nabla \overset{\circ}{h}|^2 - |\nabla |\overset{\circ}{h}|^2|^2 = |\overset{\circ}{h}|^2 |\nabla H|^2 - 2\overset{\circ}{h}_{ij} \nabla^i |\overset{\circ}{h}|^2 \nabla^j H$$

and equation (35) follows from $|\nabla \overset{\circ}{h}|^2 = |\nabla h|^2 - \frac{1}{2} |\nabla H|^2$. □

Corollary 4.3. *i) From* $g^{kl} \overset{\circ}{h}_{ik} \overset{\circ}{h}_{jl} = |\overset{\circ}{h}|^2 g_{ij}$ *it follows*

$$2|\overset{\circ}{h}|^2 |\nabla \overset{\circ}{h}|^2 = |\nabla_i |\overset{\circ}{h}|^2 - \overset{\circ}{h}_{ij} \nabla^j H|^2$$

ii) In case $\nabla H = 0$ *we obtain the optimal Kato identity*

$$(2|h|^2 - H^2) |\nabla h|^2 = |\nabla |h|^2|^2 .$$

iii) Applying (35) to the mean curvature H *and the Gauß curvature* $K = \det h^i{}_j$ *we get*

$$\begin{aligned} &(H^2 - 4K)\big(|\nabla h|^2 - |\nabla H|^2\big) \\ &\qquad = 2(H g^{ij} - h^{ij})\big(H \nabla_i H \nabla_j H - 2\nabla_i H \nabla_j K\big) \\ &\qquad\quad -2H \langle \nabla H, \nabla K \rangle + 4|\nabla K|^2 . \end{aligned} \tag{41}$$

4.1. **$\mathbf{C^2}$-estimates.** Let us first compute the evolution equation of the Gauß curvature K. We use equation (32) with $G = K$. In this case we obtain

$$G_K = 1, \quad G_H = 0, \quad G^{ij} = Hg^{ij} - h^{ij}\,, \quad G^{pq,kl} = g^{kl}g^{pq} - g^{pk}g^{ql}\,. \tag{42}$$

Then a straightforward computation using Codazzi's equation and equations (21), (41) shows

$$\begin{aligned}
&\left(G^{ij}\sigma^{pq,kl} - \sigma^{ij}G^{pq,kl}\right)\nabla_i h_{kl}\nabla_j h_{pq}\\
= \;&\frac{1}{\left(\kappa H^2 + (\kappa-K)^2\right)^2}\Big\{(\kappa^2-K^2)(H^2+2(\kappa-K))(|\nabla h|^2-|\nabla H|^2)\\
&\qquad - (\kappa^2-K^2)H(Hg^{ij}-h^{ij})\nabla_i H\nabla_j H\\
&\qquad - 2\kappa H^2(Hg^{ij}-h^{ij})\nabla_i H\nabla_j K\\
&\qquad + 2H(\kappa-K)(Hg^{ij}-h^{ij})\nabla_i K\nabla_j K\\
&\qquad - (\kappa-K)^2\left(H\langle\nabla H,\nabla K\rangle - 2|\nabla K|^2\right)\Big\}\,.
\end{aligned}$$

Inserting this into equation (32) we have shown

Lemma 4.4. *In dimension 2 there exists a smooth function S and a smooth vector field V such that the evolution equation of the Gauß curvature K induced by the flow (1) can be written in the form*

$$\frac{d}{dt}K = \sigma^{ij}\nabla_i\nabla_j K + \langle\nabla K, V\rangle + (\kappa^2-K^2)S - (\kappa+K)H\phi\,. \tag{43}$$

Lemma 4.5. *Suppose $n = 2$ and that $K^2 < \kappa^2$ at $t = 0$. Then there exists a constant $\varepsilon > 0$ such that the estimates*

$$\kappa - K > \varepsilon\,, \tag{44}$$
$$\kappa + K > 0 \tag{45}$$

hold for all $t \in [0, T)$.

Proof. For real numbers x, y with $xy < 1$ one has

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}\,.$$

If λ_1, λ_2 denote the two principal curvatures, then $\kappa - K > 0$ is equivalent to

$$\frac{\lambda_1}{\sqrt{\kappa}} \cdot \frac{\lambda_2}{\sqrt{\kappa}} < 1$$

and hence we have

$$\phi = \frac{1}{\sqrt{\kappa}}\left(\arctan\frac{\lambda_1}{\sqrt{\kappa}} + \arctan\frac{\lambda_2}{\sqrt{\kappa}}\right) = \frac{1}{\sqrt{\kappa}}\arctan\frac{\sqrt{\kappa}\,H}{\kappa - K}\,.$$

Lemma 3.6 implies that there exists a positive constant $C < \pi/2$, depending only on $\sup_{p\in M}|\phi(p,0)|$, such that

$$\kappa H^2 \le (\kappa - K)^2 \tan^2(Ce^{-2t})$$

as long as $\kappa - K \ge 0$. Adding $(\kappa - K)^2$ to both sides we get

$$\kappa^2 \le (\kappa + \lambda_1^2)(\kappa + \lambda_2^2) = \kappa H^2 + (\kappa - K)^2 \le (\kappa - K)^2\big(1 + \tan^2(Ce^{-2t})\big)$$

and therefore

$$(\kappa - K)^2 \ge \kappa^2 \cos^2(Ce^{-2t}) > 0$$

as long as $\kappa - K \ge 0$. This shows that $\kappa - K$ cannot tend to zero and that inequality (44) holds with $\varepsilon := \kappa\cos(C)$. The second inequality (45) follows from the evolution equation of K given by equation (43) and the maximum principle. $\square$

So far we have shown that the Gauß curvature K stays uniformly bounded. To prevent the formation of thin "necks" we need to control the full norm $|h|$ of the second fundamental form. But this can be achieved by exploiting the bounds of K and $\tan(\sqrt{\kappa}\phi)$ since

$$\kappa|h|^2 = \kappa H^2 - 2\kappa K = (\kappa - K)^2\tan^2(\sqrt{\kappa}\phi) - 2\kappa K$$

and the RHS is bounded.

We summarize:

Lemma 4.6. *Suppose $n = 2$ and that the Gauß curvature satisfies $|K| < \kappa$ at $t = 0$. Then the second fundamental form h stays uniformly bounded for all $t \in [0, T)$.*

4.2. **$\mathbf{C^1}$-estimates.** Once we have uniform C^2-estimates it is easy to derive uniform C^1-estimates. Since the induced Riemannian metric g_{ij} evolves according to

$$\frac{d}{dt}\,g_{ij} = 2\phi h_{ij}$$

and the second fundamental tensor h_{ij} is uniformly bounded, we conclude that there exists a uniform constant $C' > 0$ such that

$$\left|\frac{d}{dt}\,g_{ij}\right| \leq C'|\phi|g_{ij}$$

holds for all $t \in [0, T)$. By Lemma 3.6 this can be further estimated and we obtain

$$\left|\frac{d}{dt}\,g_{ij}\right| \leq Ce^{-t}g_{ij}$$

with another uniform constant $C > 0$. Thus

$$e^{-C(1-e^{-t})}g_{ij}(p, 0) \leq g_{ij}(p, t) \leq e^{C(1-e^{-t})}g_{ij}(p, 0)$$

for all $(p, t) \in M \times [0, T)$. In particular, all induced metrics are uniformly equivalent to the initial metric.

4.3. **Longtime existence and convergence.** Using the key estimates obtained in the previous subsections, we are now able to prove Theorem 1.2.

Proof of Theorem 1.2: Since we have uniform C^2-estimates we proceed as in [A] (see also [A2] for details) and use a parabolic variant of the classical Morrey and Nirenberg estimates for fully nonlinear elliptic equations in two space variables to obtain uniform $C^{2,\alpha}$-estimates in space and $C^{1,\alpha}$-estimates in time. Schauder theory then implies uniform C^k-estimates in space and time for any $k \geq 1$ and hence longtime existence of a smooth solution. Since in local coordinates we have $\frac{d}{dt}\,F^\alpha = \phi\nu^\alpha$ we compute for any $0 \leq t_1 \leq t_2 < T$

$$\begin{aligned} |F^\alpha(p, t_2) - F^\alpha(p, t_1)| &= \left|\int_{t_1}^{t_2} \phi(p, t)\nu^\alpha(p, t)dt\right| \\ &\leq C(e^{-2t_1} - e^{-2t_2}) \end{aligned} \tag{46}$$

with some constant $C > 0$ independent of t_1, t_2, where we have used Lemma 3.6 and the fact that all induced Riemannian metrics are uniformly equivalent. Thus for any $\epsilon > 0$ and any $p \in M$ there exists some $t_1 > 0$ such that for all $t_2 \geq t_1$ the points $F(p, t_2)$ and $F(p, t_1)$ lie in the same coordinate chart and the euclidean distance $|F(p, t_2) - F(p, t_1)|$ in this coordinate chart is bounded by ϵ. Since M is compact we obtain uniform and by (46) exponential convergence of $M_t = F(M, t)$ to a smooth limiting surface $M_\infty \subset N$ as $t \to \infty$. M_∞ is spacelike since all induced Riemannian metrics stay uniformly equivalent. Since by Lemma 4.5 the estimate $\kappa - K > \epsilon$ holds for some $\epsilon > 0$ and all $t \geq 0$

we can express ϕ in the form $\phi = \frac{1}{\sqrt{\kappa}} \arctan \frac{\sqrt{\kappa}H}{\kappa - K}$ for all $t \in [0, \infty)$. Then Lemma 3.6 implies that ϕ and H tend to zero as $t \to \infty$ and hence the mean curvature H of the limiting surface vanishes. Again by Lemma 4.5 we obtain that the Gauß curvature of the limiting surface M_∞ satisfies $-\kappa \le K \le \kappa - \epsilon < \kappa$. This proves Theorem 1.2. □

5. Appendix

With the same notations as before let us now assume that $F : M \to N$ is a spacelike immersion of a surface M into the anti-De Sitter manifold $N = \mathrm{AdS}_3$, represented by

$$\mathrm{AdS}_3 = \{V \in \mathbb{R}^4_2 : \langle V, V\rangle_{2,4} = -1\}$$

with its induced Lorentzian metric $\langle \cdot, \cdot \rangle_N$ that it inherits from $\mathbb{R}^4_2 = \left(\mathbb{R}^4, \langle \cdot, \cdot \rangle_{2,4}\right)$, where the inner product on $\mathbb{R}^4_2$ is given by

$$\langle V, W\rangle_{2,4} = V^1W^1 + V^2W^2 - V^3W^3 - V^4W^4 \,.$$

AdS_3 is a 3-dimensional Lorentzian space form of constant sectional curvature -1 and is a vacuum solution of Einstein's equation with cosmological constant $\Lambda < 0$. AdS_3 contains closed timelike curves and hence is not simply connected. The simply connected universal cover of AdS_3 will also be called anti-De Sitter space and we denote it $\widetilde{\mathrm{AdS}}_3$.

Thus an immersion $F : M \to \mathrm{AdS}_3$ can also be seen as an immersion of M into $\mathbb{R}^4_2$. The Gauß map of F is the map

$$\mathscr{G} : M \to Gr_2^+(2, 4)\,, \quad p \mapsto T_pM \in Gr_2^+(2, 4)\,,$$

where T_pM is considered as an oriented spacelike surface in $\mathbb{R}^4_2$ and $Gr_2^+(2, 4)$ denotes the Grassmannian of oriented spacelike surfaces in $\mathbb{R}^4_2$. It is well known that the Grassmannian $Gr_2^+(2, 4)$ is isometric to $\mathbb{H}_{1/\sqrt{2}} \times \mathbb{H}_{1/\sqrt{2}}$, where $\mathbb{H}_{1/\sqrt{2}}$ denotes the scaled hyperbolic plane

$$\mathbb{H}_{1/\sqrt{2}} := \left\{ V \in \mathbb{R}^3_1 : \langle V, V\rangle_{1,3} = -\frac{1}{2}\,, V^3 > 0 \right\}\,,$$

where $\left(\mathbb{R}^3_1, \langle \cdot, \cdot\rangle_{1,3}\right)$ denotes the usual Minkowski space. To understand the geometry of the Gauß map $\mathscr{G}$ it is convenient to use the isometry between $Gr_2^+(2, 4)$ and $\mathbb{H}_{1/\sqrt{2}} \times \mathbb{H}_{1/\sqrt{2}}$ (since the sectional curvature of $\mathbb{H}_{1/\sqrt{2}}$ is -2, $Gr_2^+(2, 4)$ is a Kähler-Einstein manifold of scalar curvature $S = -8$).

Let e_1, e_2, e_3, e_4 denote the standard basis of $\mathbb{R}^4$. We introduce a set of endomorphisms on $\mathbb{R}^4$:

$$\begin{aligned}
&E^1_+ : \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} e_4 \\ -e_3 \\ -e_2 \\ e_1 \end{pmatrix}, \qquad && E^1_- : \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} -e_4 \\ -e_3 \\ -e_2 \\ -e_1 \end{pmatrix}, \\
&E^2_+ : \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} -e_3 \\ -e_4 \\ -e_1 \\ -e_2 \end{pmatrix}, \qquad && E^2_- : \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} -e_3 \\ e_4 \\ -e_1 \\ e_2 \end{pmatrix}, \\
&E^3_+ : \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} e_2 \\ -e_1 \\ -e_4 \\ e_3 \end{pmatrix}, \qquad && E^3_- : \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} e_2 \\ -e_1 \\ e_4 \\ -e_3 \end{pmatrix},
\end{aligned} \tag{47}$$

These endomorphisms satisfy

$$E^1_+E^2_+ = -E^2_+E^1_+ = E^3_+\,, \quad (E^1_+)^2 = (E^2_+)^2 = -(E^3_+)^2 = \mathrm{Id}\,,$$
$$E^1_-E^2_- = -E^2_-E^1_- = E^3_-\,, \quad (E^1_-)^2 = (E^2_-)^2 = -(E^3_-)^2 = \mathrm{Id}$$

Moreover, if $V \in \mathbb{R}^4$ is an arbitrary nonzero vector, then

$$\{V, E^1_+V, E^2_+V, E^3_+V\}$$

forms a positively oriented basis and

$$\{V, E^1_-V, E^2_-V, E^3_-V\}$$

a negatively oriented basis of $\mathbb{R}^4$. Associated to these endomorphisms are the following six symplectic 2-forms:

$$\omega^A_+ := \left\langle E^A_+\cdot, \cdot\right\rangle_{2,4} \quad \text{and} \quad \omega^A_- := \left\langle E^A_-\cdot, \cdot\right\rangle_{2,4}\,, \quad A = 1, 2, 3$$

and we have

$$\begin{aligned}
\omega^1_+ &= e_2 \wedge e_3 + e_4 \wedge e_1, \qquad & \omega^1_- &= e_2 \wedge e_3 - e_4 \wedge e_1, \\
\omega^2_+ &= e_1 \wedge e_3 + e_2 \wedge e_4, \qquad & \omega^2_- &= e_1 \wedge e_3 - e_2 \wedge e_4, \\
\omega^3_+ &= e_1 \wedge e_2 + e_3 \wedge e_4, \qquad & \omega^3_- &= e_1 \wedge e_2 - e_3 \wedge e_4.
\end{aligned}$$

For any spacelike unit vector e and any two vectors V, W we have

$$\begin{aligned} \langle V, W\rangle_{2,4} &= -\omega_+^1(e,V)\omega_+^1(e,W) - \omega_+^2(e,V)\omega_+^2(e,W) \\ &\quad +\omega_+^3(e,V)\omega_+^3(e,W) + \langle e, V\rangle_{2,4}\langle e, W\rangle_{2,4} \\ &= -\omega_-^1(e,V)\omega_-^1(e,W) - \omega_-^2(e,V)\omega_-^2(e,W) \\ &\quad +\omega_-^3(e,V)\omega_-^3(e,W) + \langle e, V\rangle_{2,4}\langle e, W\rangle_{2,4} . \end{aligned} \tag{48}$$

Let ν_N denote the future directed timelike unit normal along $\mathrm{AdS}_3 \subset \mathbb{R}^4_2$. The orientation of $\mathbb{R}^4$ induces an orientation on AdS_3 in the following way: We say that $V_1, V_2, V_3 \in T_q\mathrm{AdS}_3$ is positively oriented, if V_1, V_2, V_3, ν_N represents the positive orientation of $\mathbb{R}^4$.

Now let $F : M \to \mathrm{AdS}_3$ be a spacelike immersion of an oriented surface and let ν denote the future directed timelike unit normal of M within AdS_3. We will assume that the orientation of M is chosen in such a way that for any positively oriented basis $\{e_1, e_2\}$ of T_pM the basis $\{DF(e_1), DF(e_2), \nu\}$ represents the positive orientation of $T_{F(p)}\mathrm{AdS}_3$.

For such an immersion let us define the six functions

$$\mathscr{G}_+^A : M \to \mathbb{R}\,, \quad \mathscr{G}_+^A := \frac{1}{\sqrt{2}} *\!\big(F^*\omega_+^A\big)\ , \quad A = 1, 2, 3$$

and

$$\mathscr{G}_-^A : M \to \mathbb{R}\,, \quad \mathscr{G}_-^A := \frac{1}{\sqrt{2}} *\!\big(F^*\omega_-^A\big)\ , \quad A = 1, 2, 3\,,$$

where F^* denotes "pull-back" and $*$ is the Hodge-Operator on forms. From equation (48) one immediately gets [1)]

$$(\mathscr{G}_+^1)^2 + (\mathscr{G}_+^2)^2 - (\mathscr{G}_+^3)^2 = -\frac{1}{2} = (\mathscr{G}_-^1)^2 + (\mathscr{G}_-^2)^2 - (\mathscr{G}_-^3)^2 \tag{49}$$

and by construction we have $\mathscr{G}_\pm^3 > 0$, so that

$$\mathscr{G}_+ = (\mathscr{G}_+^1, \mathscr{G}_+^2, \mathscr{G}_+^3) \quad \text{and} \quad \mathscr{G}_- = (\mathscr{G}_-^1, \mathscr{G}_-^2, \mathscr{G}_-^3)$$

define two functions from M to $\mathbb{H}_{1/\sqrt{2}}$. $\mathscr{G}_+, \mathscr{G}_-$ are called the self-dual resp. the anti-self-dual Gauß maps of F and the Gauß map $\mathscr{G} : M \to Gr_2^+(2,4) = \mathbb{H}_{1/\sqrt{2}} \times \mathbb{H}_{1/\sqrt{2}}$ is given by the pair $\mathscr{G} = (\mathscr{G}_+, \mathscr{G}_-)$.

Let

$$\langle\cdot,\cdot\rangle_{Gr_2^+(2,4)}\,, \quad \mathscr{J} \quad \text{and} \quad \omega = \langle \mathscr{J}\,\cdot, \cdot\rangle_{Gr_2^+(2,4)}$$

[1)] Choose an arbitrary unit tangent vector e of T_pM and let $V = W = Ce$, where C denotes the complex structure on M induced by the orientation and Riemannian metric on M

denote the Kähler metric, complex structure and Kähler form on the Grassmannian $Gr_2^+(2,4) = \mathbb{H}_{1/\sqrt{2}} \times \mathbb{H}_{1/\sqrt{2}}$.

As was shown in [T], [TU], we have $\mathscr{G}^*\omega = 0$, i.e. the Gauß map of an immersion $F : M \to \mathrm{AdS}_3$ defines a Lagrangian immersion $\mathscr{G} : M \to Gr_2^+(2,4)$.

Let

$$\sigma_{ij} dx^i \otimes dx^j = \mathscr{G}^* \langle \cdot, \cdot \rangle_{Gr_2^+(2,4)}$$

denote the Riemannian metric on M induced by the Gauß map. If D denotes the connection associated to σ, then it is well known that the second fundamental tensor

$$\tau_{ijk} = \omega(D_i \mathscr{G}, D_j D_k \mathscr{G})$$

of the Lagrangian immersion is completely symmetric and that the mean curvature form $\tau = \tau_i dx^i$ on M, i.e. its trace $\tau_i = \sigma^{jk}\tau_{ijk}$ (where $(\sigma^{jk})_{j,k=1,2}$ denotes the inverse of $(\sigma_{jk})_{j,k=1,2}$), is closed.

The first and second fundamental forms on M induced by F shall be denoted (as before) by $g_{ij} dx^i \otimes dx^j$ and $h_{ij} dx^i \otimes dx^j$. If we consider F as a map from M to $\mathbb{R}^4_2$, then the Gauß formula shows that the second fundamental tensor $\tilde{A}$ of M, considered as a submanifold of codimension two in $\mathbb{R}^4_2$, decomposes into

$$\tilde{A}_{ij} = g_{ij}\nu_N + h_{ij}\nu\,. \tag{50}$$

Lemma 5.1. *Let $F : M \to \mathrm{AdS}_3$ be a spacelike immersion. With the same notations as above the following relations between the first and second fundamental forms of F and $\mathscr{G}$ are valid:*

$$\begin{aligned} \sigma_{ij} &= g_{ij} + g^{kl} h_{ik} h_{jl}\,, & (51)\\ \tau_{ijk} &= \nabla_i h_{jk}\,, & (52) \end{aligned}$$

where ∇ denotes the Levi-Civita connection of g_{ij}.

Proof. Straightforward computations using (48), (49) and (50). □

In particular, we observe that σ_{ij} coincides with the tensor defined earlier in equation (18) since in this special situation we have $\kappa = 1$. As a corollary we obtain:

Lemma 5.2. *The Maslov class of the Gauß map $\mathscr{G}$ is trivial and the Lagrangian angle is given by $\phi = \arctan\lambda_1 + \arctan\lambda_2$, where λ_1, λ_2 are the principal curvatures of $F : M \to \mathrm{AdS}_3$.*

Proof. We have seen in Lemma 3.4 that $\phi = \arctan\lambda_1 + \arctan\lambda_2$ is a smooth function. Moreover we have $\frac{\partial\phi}{\partial h_{ij}} = \sigma^{ij}$ and then

$$\begin{aligned}
\nabla_k\phi &= \frac{\partial\phi}{\partial h_{ij}}\nabla_k h_{ij} + \frac{\partial\phi}{\partial g^{ij}}\nabla_k g^{ij} \\
&= \sigma^{ij}\nabla_k h_{ij} \\
&\overset{(52)}{=} \sigma^{ij}\tau_{kij} \\
&= \tau_k\,.
\end{aligned}$$

This means that the mean curvature form τ of the Gauß map satisfies $\tau = d\phi$. Since τ/π represents the Maslov class, it must be trivial. $\square$

We will now treat the case where $F : M \times [0,T) \to \mathrm{AdS}_3$ is a smooth family of spacelike immersions satisfying an evolution equation of the form

$$\frac{d}{dt}F = f\nu\,,$$

where f is an arbitrary smooth function and ν the future directed timelike unit normal. The Gauß maps of F depend on t and will vary in time. A straightforward computation gives the two relations

$$\left\langle \frac{d}{dt}\mathscr{G}, D_k\mathscr{G}\right\rangle_{Gr_2^+(2,4)} = g^{ml}h_{lk}\nabla_m f$$

and

$$\left\langle \frac{d}{dt}\mathscr{G}, \mathscr{J} D_k\mathscr{G}\right\rangle_{Gr_2^+(2,4)} = \nabla_k f\,,$$

so that

$$\frac{d}{dt}\mathscr{G} = \mathscr{J}\left(\sigma^{kl}\nabla_k f D_l\mathscr{G}\right) + \sigma^{kl}g^{ms}h_{sk}D_m f D_l\mathscr{G}\,.$$

So we have shown:

Lemma 5.3. *Suppose $F : M \times [0,T) \to \mathrm{AdS}_3$ is a smooth family of spacelike immersions driven by the flow $\frac{d}{dt}F = f\nu$, where ν denotes the future directed timelike unit normal. Then the Gauß maps $\mathscr{G}_F : M \times [0,T) \to Gr_2^+(2,4)$ of F evolve according to*

$$\frac{d}{dt}\mathscr{G}_F = \left(\mathscr{J}\circ d\mathscr{G}_F + d\mathscr{G}_F\circ\mathscr{W}_F\right)\nabla^\sigma f\,. \tag{53}$$

where $\mathscr{J}$ denotes the complex structure on $Gr_2^+(2,4)$, $\mathscr{W}_F$ is the Weingarten map of F and $\nabla^\sigma f$ denotes the gradient of f w.r.t. the induced metric $\sigma = \mathscr{G}^*\langle\cdot,\cdot\rangle_{Gr_2^+(2,4)}$.

In particular, if we choose for f the Lagrangian angle $\phi = \arctan\lambda_1 + \arctan\lambda_2$, then - up to the tangential term $(d\mathscr{G}_F \circ \mathscr{W}_F)\nabla^\sigma f$, which is of no interest concerning the geometric evolution - the Gauß maps evolve by the Lagrangian mean curvature flow.

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